Tree induction

☆ note how we split the large tree of height k into smaller subtrees of height k-1 or shorter ☆

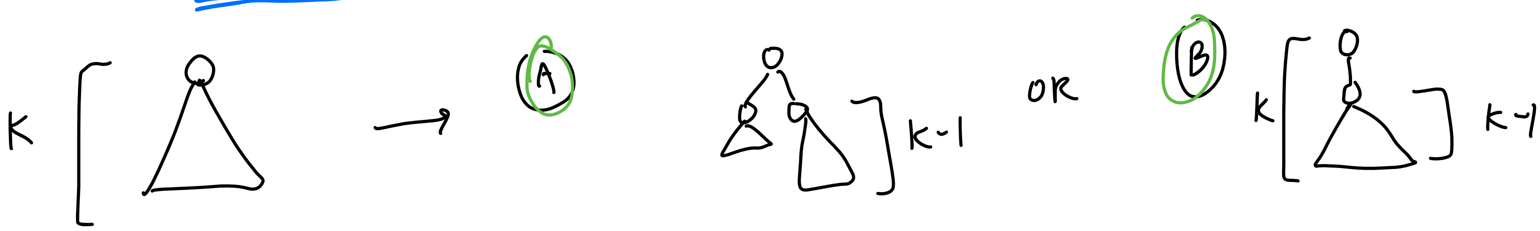
tree example) Let T be a binary tree with height h and n-many nodes.
Then, $n \le 2^{h+1} - 1$.

Proof by induction on height h.

Base: $h = 0$, $n = 1$ $1 \le 2^1 - 1 = 1$ ✓

IH: Suppose binary tree of height h & n-many nodes has $n \le 2^{h+1} - 1$ for $h = 1 \ldots k-1$.

Goal: for a tree of height k, it's true that $n \le 2^{k+1} - 1$.



Inductive step:

case 1: Assume we have a tree of height k. case 1 is that the root node has 2 children. Then, each of those children is the root node of a tree of height $\le k-1$. The left subtree then has $\le 2^k - 1$ nodes, by IH. The right subtree also has $\le 2^k - 1$ nodes, by IH.

Then our tree of height k has $\le \cancel{1} + 2^k \cancel{-1} + 2^k - 1$ nodes.

$$n \le 2 \cdot 2^k - 1 = 2^{k+1} - 1$$

case 2: Assume we have a tree of height k. Case 2 is that the root node hase 1 child of height = k-1. By IH, this subtree has $\leq 2^k - 1$ nodes. So our tree has $n \leq 2^k - 1 + 1 = 2^k \leq 2^{k+1} - 1$ since $k > 0$.

Grammar example)

consider grammar G with start S and terminals a, b.

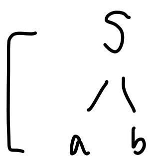
base case

$S \to ab \mid SS \mid aSb$.

Claim: All trees generated by G will have the same number of a nodes and b nodes.

Proof by induction on height h.

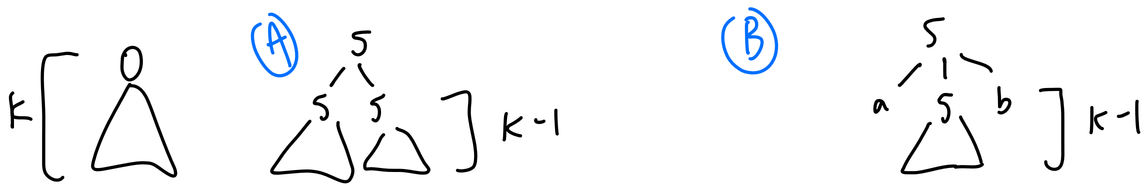
Base: ~~h=0~~ S h=1



The number of a's and b's are equal (1).

IH: Suppose all trees generated by G of height h will have the same # of a and b nodes for h = 1...k-1.

Goal: Show tree of height k generated by G has equal # a's & b's.



case 1: Our tree of height k has 2 children rooted at S, call the subtrees S_1 and S_2. Both S_1 and S_2 have height $\leq k-1$. By IH, S_1 has m-many a's and b's, and S_2 has n-many a's and b's, $m, n \in \mathbb{N}$. So, our tree has m+n-many a's and m+n-many b's.

Case 2: Our tree of height k has 3 children, rooted at $a\,S\,b$.
The subtree rooted at S has n-many a's and b's.
Our tree then has $n+1$-many a's and b's from the left & right children.